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Translated by N. H. C.

UDC 62-50

STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH NONCONVEX CONSTRAINTS

PMM Vol. 38, №3, 1974, pp. 409-416
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(Received October 22, 1973)

We consider problems of the optimal control of a linear system subject to random actions. We assume that the system's phase coordinates are connected by nonconvex constraints which are necessarily also stochastic. We discuss deterministic problems equivalent to the stochastic ones mentioned. The unified approach to the problems formulated is based on the method for solving linear control systems, developed in [1] and modified in [2 – 4] for systems with constraints.

1. Let there be a control system

$$dx / dt = A(t)\; x(t) + B(t)\; u(t) + \xi(t) \tag{1.1}$$

where $x(t)$ is the n-dimensional phase coordinate vector, u is the r-dimensional control vector, $A(t)$, $B(t)$ are known continuous matrices of appropriate dimension, $\xi(t)$ is an n-dimensional vector-valued random process with specified probabilistic characteristics. The deterministic controls $u(t)$ (elements $u(\cdot)$) are chosen from a fixed weakly-compact convex set U of functions $u(t)$ from the r-vector space $L_2[t_\alpha,\ t_\beta]$. The control of the deterministic component

$$QX[t,\ t_\alpha]\, x^{(\alpha)} + Q \int_{t_\alpha}^{t} H[t, \tau]\, u(\tau)\, d\tau = y(t) \tag{1.2}$$

of the state vector $x(t)$ of system (1.1) is effected by choosing $u(\cdot) \in U$.

Problem 1.1. Given an initial state $x(t_\alpha) = x^{(\alpha)}$, a point $x^{(\beta)}$, a number $\varepsilon > 0$, a continuous function $\nu(t) > 0$ and an n-vector-valued function $x^\circ(t)$. From among the controls $u(\cdot) \in U$ find the $u^\circ(t)$ satisfying the condition $t_\beta{}^\circ - t_\alpha = \min$ under the constraints

$$M\rho_1[P\,(x(t_\beta) - x^{(\beta)})] \leqslant \varepsilon \tag{1.3}$$

$$M\rho_2[Q\,(x(t) - x^\circ(t))] \geqslant \nu(t), \quad t_\alpha \leqslant t \leqslant t_\beta \tag{1.4}$$

Here P, Q are known matrices of dimension $p \times n$, $q \times n$, respectively, $\rho_1[z_1]$, $\rho_2[z_2]$ are nonnegative convex functions in the spaces $R^{(p)}$, $R^{(q)}$ $(p,\ q \leqslant n)$, for which we can find numbers $\alpha > 0,\ k \geqslant 1$ so as to satisfy the conditions

$$\rho_i[z_i] \leqslant \alpha[1 + \|z_i\|^k], \quad i = 1, 2 \tag{1.5}$$

($\| z \|$ is the Euclidean norm, M is the symbol for the mean).

We can formulate as well a problem reciprocal to Problem 1.1.

Problem 1.2. On a given interval $[t_\alpha, t_\beta]$ find the control $u^0(t) \in \dot{U}$ giving the condition $\varepsilon^0 = \min \{\varepsilon\}$ under the constraints $x(t_\alpha) = x^{(\alpha)}$, (1.3), (1.4).

The problems formulated contain the nonconvex probabilistic constraints (1.4) on the phase coordinates and the convex probabilistic constraints (1.3) on the system's terminal states. In particular, Problem 1.1 is, in fact, that of the shortest time to take the system from the position $x^{(\alpha)} \in R^{(n)}$ to a probabilistic neighborhood of the point $z^{(\beta)} = Px^{(\beta)} \in R^{(p)}$ in such a way that the phase constraint (1.4) is satisfied at each instant.

2. Let us describe the solution of Problem 1.1, setting $x^0(t) \equiv 0$. A consideration of the general case does not bring in any new essential aspects in the arguments. Let U_1 be the set of controls $u(\cdot) \in L_2[t_\alpha, t_\beta]$ taking system (1.1) from the point $x(t_\alpha) = x^{(\alpha)}$ onto the manifold $\{z\} \subset R^{(p)}$ so that $M\rho_1[P\eta(t_\beta) - z] \leqslant \varepsilon$. Here the random variable $\eta(t_\beta)$ and the deterministic vector z are connected by the relations

$$\eta(t_\beta) = \int_{t_\alpha}^{t_\beta} X[t_\beta, \tau]\, \xi(\tau)\, d\tau \tag{2.1}$$

$$z^{(\beta)} - z + P\eta(t_\beta) = Px(t_\beta) \tag{2.2}$$

It can be shown that set U_1 is convex and weakly closed. Then, the set $U^* = U \cap U_1$ is convex and weakly closed.

We denote the support functional of set $C \subset L_2$ by $\rho(h(\cdot) \mid C)$. On the basis of the generalized Hahn-Banach theorem [5] we obtain an expression for the support functional $\rho(h(\cdot) \mid U^*)$ of set U^*

$$\begin{gathered}\rho(h(\cdot) \mid U^*) = \inf \{\rho(h(\cdot) + p'PH[t_\beta, \cdot] \mid U) - (p \cdot z^{(\beta)}) + \\ \rho(p \mid N) + p'PX[t_\beta, t_\alpha]x^{(\alpha)}\} \quad \text{over all } p \in R^{(p)}\end{gathered} \tag{2.3}$$

$$H[t_\beta, t] = X[t_\beta, t]B(t), \qquad N = \{z \mid M\rho_1[P\eta(t_\beta) - z] \leqslant \varepsilon\}$$

Here $X[t, t_\alpha]$ is the normed fundamental matrix of system (1.1), $(a \cdot b)$ is the scalar product of vectors a and b.

We consider a set P of deterministic vector-valued functions $y(t)$ corresponding to the controls $u(\cdot) \in U^*$ such that

$$\rho_3[y(t)] = M\rho_2[y(t) + Q\eta(t)] \geqslant \nu(t), \qquad t_\alpha \leqslant t \leqslant t_\beta \tag{2.4}$$

Here $\eta(t)$, $y(t)$ have been defined by equalities (1.2), (2.1). We can verify that $\rho_3[y]$ is a nonnegative closed convex function [6] in $R^{(q)}$. We assume that the random process $\xi(t)$ has continuous kth-order moments. Then $\rho_3[y(t)]$ is everywhere a finite function for every $t \in [t_\alpha, t_\beta]$. From the properties of convex functions it follows that $\rho_3[y(t)]$ can be represented in the form

$$\rho_3[y(t)] = \max_{(l, \mu^*) \in F^*} \{(l \cdot y(t)) - \mu^*\} = (l(t) \cdot y(t)) - \rho_3^*[l(t)], \quad t_\alpha \leqslant t \leqslant t^\beta \tag{2.5}$$

where F^* is the epigraph of the function $\rho_3^*[l]$ adjoint to $\rho_3[y]$. The maximum in formula is indeed achieved since the function $(l \cdot y(t)) - \mu^*$ and the set $F^* =$ epi $\rho_3^*[l]$ do not have common directions of recession (see [6]).

By W we denote the set of vector-valued functions $l(t)$ giving the maximum in (2.5) for the realizations $y(\cdot) \in P$. In what follows we shall assume that the epigraph of function $\rho_3[y]$ is a smooth set. Then, the vector-valued functions $l(\cdot) \in W$ are continuous. We assume, moreover, that the set W forms a compactum in space $C^{(q)}$. We consider the following auxiliary problem.

Problem 2.1. Find the control $u(\cdot) \in U^*$, $t \in [t_\alpha, t_\beta]$, ensuring the fulfillment of condition (2.4).

It is clear that the auxiliary problem has a solution if and only if there exists $l(\cdot) \in W$ satisfying the inequality

$$(l(t) \cdot y(t)) - \rho_3^*[l(t)] \geqslant \nu(t), \quad t \in [t_\alpha, t_\beta] \tag{2.6}$$

for any control $u(\cdot) \in U^*$. Then the condition for the solvability of the system of inequalities (2.6) for some $l(\cdot) \in W$ is equivalent to the condition for the solvability of the generalized moment problem

$$\int_{t_\alpha}^{t} l'(t) QH[t, \tau] u(\tau) d\tau \geqslant l'(t) QX[t, t_\alpha] x^{(\alpha)} + \rho_3^*[l(t)] + \nu(t) \tag{2.7}$$

$$\int_{t_\alpha}^{t_\beta} h'(t) u(t) dt \leqslant \rho(h(\cdot) | U) \qquad \text{for all} \quad h(\cdot) \in L_2 \tag{2.8}$$

Applying the procedure described in [2 – 4] and passing to the Stieltjes integral, the necessary and sufficient condition for the solvability of problem (2.7), (2.8) is written in the form

$$\min_{\Lambda(t)} \Big\{ \rho\Big(\int_{\tau}^{t_\beta} l'(t) QH[t, \tau] d\Lambda(t) \,|\, U^*\Big) + \tag{2.9}$$
$$\int_{t_\alpha}^{t_\beta} l'(t) QX[t, t_\alpha] x^{(\alpha)} d\Lambda(t) - \int_{t_\alpha}^{t_\beta} \nu(t) d\Lambda(t) - \int_{t_\alpha}^{t_\beta} \rho_3^*[l(t)] d\Lambda(t) \Big\} \geqslant 0$$

Here the minimum is taken over all nondecreasing functions of unit variation. Then, making use of formula (2.3), we get that the required necessary and sufficient condition for the solvability of the auxiliary Problem 2.1 is the fulfillment of the inequality

$$\max_{l(t)} \min_{\Lambda(t)} \min_{p} \Big\{ \rho\Big(\int_{\tau}^{t_\beta} l'(t) QH[t, \tau] d\Lambda(t) + p'PH[t_\beta \tau] \,|\, U \Big) + \tag{2.10}$$
$$\rho(p | N) - (p \cdot z^{(\beta)}) + p'PX[t_\beta, t_\alpha] x^{(\alpha)} +$$
$$\int_{t_\alpha}^{t_\beta} l'(t) QX[t, t_\alpha] x^{(\alpha)} d\Lambda(t) - \int_{t_\alpha}^{t_\beta} \nu(t) d\Lambda(t) - \int_{t_\alpha}^{t_\beta} \rho_3^*[l(t)] d\Lambda(t) \Big\} \geqslant 0$$

over all $l(\cdot) \in W$, $\| p \| + \mathrm{Var}\, \Lambda(t) = 1$.

We note that $\rho_3^*[l]$ is continuous on set W; therefore, the last integral in (2.9), (2.10) has meaning.

Passing to the adjoint system, we can rewrite condition (2.10) as

$$\max_{l(t)} \min_{\Lambda(t)} \min_p \Big\{\rho(s(\cdot) B(\cdot) \mid U) + \rho(p \mid N) - (p \cdot z^{(\beta)}) + (s(t_\beta) \cdot x^{(\alpha)}) - \int_{t_\alpha}^{t_\beta} \nu(t)\, d\Lambda(t) - \int_{t_\alpha}^{t_\beta} \rho_3^* [l(t)]\, d\Lambda(t)\Big\} \geqslant 0 \tag{2.11}$$

over all $l(\cdot) \in W$, $\| p \| + \mathbf{Var}\, \Lambda(t) = 1$, where $s(\tau)$ is the solution of the adjoint system in the distributions

$$\frac{ds(\tau)}{d\tau} = - s(\tau) A(\tau) - l'(\tau) Q \frac{d\Lambda(\tau)}{d\tau} \tag{2.12}$$

with boundary conditions $s(t_\beta) = p'P$ ($d\Lambda(t) / dt$ is the generalized derivative of the function $\Lambda(t)$).

Let t_β° be the smallest instant for which inequality (2.11) is fulfilled, and let $l_0(t)$, $\Lambda^\circ(t)$, p_0 be the extremal elements of (2.11), i.e.

$$\rho(s^\circ(\cdot) B(\cdot) \mid U) + \rho(p_0 \mid N) - (p_0 \cdot z^{(\beta)}) + (s^\circ(t_\beta) x^{(\alpha)}) - \int_{t_\alpha}^{t_\beta^\circ} \nu(t)\, d\Lambda^\circ(t) - \int_{t_\alpha}^{t_\beta^\circ} \rho_3^* [l_0(t)]\, d\Lambda^\circ(t) = 0 \tag{2.13}$$

Then $t_\beta^\circ - t_\alpha$ is the optimal time for Problem 1.1. Here the optimal control $u^\circ(t)$ satisfies the maximum principle

$$\int_{t_\alpha}^{t_\beta^\circ} s^\circ(\tau) B(\tau) u^\circ(\tau)\, d\tau = \max_{u(\cdot) \in U} \int_{t_\alpha}^{t_\beta^\circ} s^\circ(\tau) B(\tau) u(\tau)\, d\tau \tag{2.14}$$

while the deterministic component $y^\circ(t)$ of the optimal trajectory satisfies the minimum principle

$$0 = \int_{t_\alpha}^{t_\beta^\circ} (l_0'(t) y^\circ(t) - \rho_3^* [l_0(t)] - \nu(t))\, d\Lambda^\circ(t)) = \min_{\rho_3 [y(t)] \geqslant \nu(t)} \int_{t_\alpha}^{t_\beta^\circ} (\rho_3 [y(t)] - \nu(t))\, d\Lambda^\circ(t) \tag{2.15}$$

The optimal aiming point z° satisfies the maximum condition

$$(p_0 \cdot z^\circ) = \max_{z \in N} (p_0 \cdot z) \tag{2.16}$$

The assertions made above can be consolidated into the following theorem.

Theorem. The necessary and sufficient condition for the solvability of Problem 1.1 is the fulfillment of condition (2.10) or (2.11). The optimal control $u^\circ(t)$ satisfies the maximum principle (2.14). The deterministic component $y^\circ(t)$ of the optimal trajectory satisfies the minimum condition (2.15), and the aiming point z° satisfies condition (2.16).

Notes. 1°. If epi $\rho_3 [y]$ is not a smooth set, then the $l(t)$ in (2.5) can turn out to be discontinuous. Then the integrals of form

$$\int_{t_\alpha}^{t_\beta} l'(t) QH [t_\beta, t]\, d\Lambda(t)$$

must be undestood in the Radon sense [7].

2°. Under the condition

$$\rho_2 [y (t), + Q\eta (t)] = \| y (t) + Q\eta (t) \|^2$$

the function $\rho_3 [y (t)]$ takes the form

$$\rho_3 [y (t)] = M\rho_2 [y (t) + Q\eta (t)] = \| y (t) + QM\eta (t) \|^2 + M\| Q\eta (t) \|^2$$

for every $t \in [t_\alpha, t_\beta]$, while the adjoint function

$$\rho_3^* [l (t)] = {}^1/_4 \| l (t) \|^2 - l' (t) \cdot QM\eta (t) - M \| Q\eta (t) \|^2$$

In particular, if $MQ\eta (t) \equiv 0$, then

$$\rho_3[y (t)] = \| y (t) \|^2 + M \| Q\eta (t) \|^2$$
$$\rho_3^* [l (t)] = {}^1/_4 \| l (t) \|^2 - M \| Q\eta (t) \|^2$$

3°. By applying arguments analogous to those in [4] we can show that if $\rho_3 [y^\circ(t)] > \nu (t)$ for $t \in [t_1, t_2] \subset [t_\alpha, t_\beta]$, then the condition $\Lambda^\circ (t) \equiv \text{const}$ for $t \in [t_1, t_2]$ is fulfilled for the extremal element $\Lambda^\circ (t)$.

4°. If the epigraph of function $\rho_3 [y]$ is a smooth set and $\Lambda^\circ (t)$ has a jump at $t = t_1 \in [t_\alpha, t_\beta]$, then the condition

$$l_0' (t_1) QB (u^{\cdot\circ} (t_1 + 0) - u^{\cdot\circ} (t_1 - 0)) \leqslant 0 \tag{2.17}$$

is necessarily satisfied. In the case $\rho_3 [y] = \| y\|^2$ the last condition can be rewritten as

$$y^\circ(t_1) (\dot{y}^\circ (t_1 + 0) - \dot{y}^\circ (t_1 - 0)) \leqslant 0 \tag{2.18}$$

Conditions (2.17), (2.18) enable us to pick out the points at which the function $\Lambda^\circ (t)$ is suspected of having a jump.

Problem 1.2, being reciprocal to Problem 1.1, can be solved from the condition of solvability of the auxiliary problem.

3. Let us discuss the solution of Problem 1.1, assuming that

$$\begin{aligned} \rho_1[z_1] &= (z_1 \cdot z_1) \\ \rho_2[z_2(t)] &= (z_2(t) \cdot z_2(t)), \quad t_\alpha \leqslant t \leqslant t_\beta \end{aligned} \tag{3.1}$$

In the given case inequalities (1.3), (1.4) are rewritten as

$$\begin{aligned} &\| P (x (t_\beta) - x^{(\beta)}) \|^2 + \sigma_1^2 \leqslant \varepsilon \\ &\| Q\bar{x} (t) \|^2 + \sigma_2^2 (t) \geqslant \nu (t), \quad t_\alpha \leqslant t \leqslant t_\beta \end{aligned} \tag{3.2}$$

$$\bar{x} (t) = X [t, t_\alpha] x^{(\alpha)} + \int_{t_\alpha}^{t} H [t, \tau] u (\tau) d\tau + \int_{t_\alpha}^{t} X [t, \tau] M\xi (\tau) d\tau$$

$$\sigma_1^2 = M \| P\eta (t_\beta) - PM\eta (t_\beta) \|^2, \quad \sigma_2^2(t) = M \| Q\eta (t) - MQ\eta (t) \|^2$$

Here σ_1^2 is the trace of the covariance matrix of the random vector $P\eta (t_\beta)$, $\sigma_2^2(t)$ is the trace of the covariance matrix of the random process $Q\eta (t)$ [8]. We introduce into consideration the deterministic system

$$\dot{\vartheta} (t) = A (t) \vartheta (t) + B (t)u (t) + \zeta (t), \quad \zeta (t) = M\xi (t)$$

and formulate the following problem.

Problem 3.1. Given the initial position $\vartheta(t_\alpha) = x^{(\alpha)}$, the point $x^{(\beta)}$, the number ε_1, and the continuous function $\nu_1(t) > 0$. Find the control $u^\circ(t) \in U$ rendering the condition $t_\beta{}^\circ - t_\alpha = \min$ under the constraints

$$\| P(\vartheta(t_\beta) - x^{(\beta)}) \|^2 \leqslant \varepsilon_1, \quad \| Q\vartheta(t) \|^2 \geqslant \nu_1(t), \quad t_\alpha \leqslant t \leqslant t_\beta$$

From (3.2) we see that under condition (3.1) the solution of Problem 1.1 can be obtained as the solution of Problem 3.1 if we assume $\varepsilon_1 = \varepsilon - \sigma_1{}^2$, $\nu_1(t) = \nu(t) - \sigma_2{}^2(t)$. Thus, Problem 1.1 is reduced to the deterministic Problem 3.1. The necessary and sufficient condition for the solvability of these problems is the fulfillment of the inequality

$$\max_{l(t)} \min_{\Lambda(t)} \min_p \Big\{ \rho \Big(\int_\tau^{t_\beta} l'(t)\, QH[t,\tau]\, d\Lambda(t) + p'PH[t_\beta,\tau] \,\Big|\, U \Big) + \quad (3.3)$$
$$\rho(p \mid N) - (p \cdot z^{(\beta)}) + p'PX[t_\beta, t_\alpha]\, x^{(\alpha)} +$$
$$\int_{t_\alpha}^{t_\beta} l'(t)\, QX[t, t_\alpha]\, x^{(\alpha)} d\Lambda(t) + \int_{t_\alpha}^{t_\beta} l'(t)\, QM\eta(t)\, d\Lambda(t) -$$
$$\int_{t_\alpha}^{t_\beta} \nu_1(t)\, d\Lambda(t) - \frac{1}{4} \int_{t_\alpha}^{t_\beta} l'(t)\, l(t)\, d\Lambda(t) \Big\} \geqslant 0$$

over all $l(\cdot) \in W$, $\| p \| + \operatorname{Var} \Lambda(t) = 1$.

This condition can also be obtained from inequality (2.10) with

$$\rho_3[y(t)] = \| y(t) + QM\eta(t) \|^2 + \sigma_2{}^2(t)$$

Passing to the adjoint system (2.12), condition (3.3) can be rewritten as

$$\max_{l(t)} \min_{\Lambda(t)} \min_p \Big\{ \rho(s(\cdot)B(\cdot) \mid U) + \rho(p \mid N) - (p \cdot z^{(\beta)}) + \quad (3.4)$$
$$(s(t_\alpha) \cdot x^{(\alpha)}) - \int_{t_\alpha}^{t_\beta} \nu_1(t)\, d\Lambda(t) - \frac{1}{4} \int_{t_\alpha}^{t_\beta} l'(t)\, l(t)\, d\Lambda(t) +$$
$$\int_{t_\alpha}^{t_\beta} l'(t)\, QM\eta(t)\, d\Lambda(t) \Big\} \geqslant 0$$

over all $l(\cdot) \in W$, $\| p \| + \operatorname{Var} \Lambda(t) = 1$.

As usual the optimal control $u^\circ(t)$ is determined from the maximum principle (2.14) on the solution $s^\circ(t)$ of the adjoint system (2.12), rendering the extremum of functional (3.4). The optimal trajectory $\vartheta^\circ(t)$ satisfies the minimum condition

$$0 = \int_{t_\alpha}^{t_\beta{}^\circ} (l_0'(t)\, Q\vartheta^\circ(t) - \frac{1}{4} l_0'(t)\, l_0(t) - \nu_1(t))\, d\Lambda^\circ(t) =$$

$$\min_{\| Q\eta(t) \|^2 \geqslant \nu_1(t)} \int_{t_\alpha}^{t_\beta{}^\circ} (\| Q\vartheta(t) \|^2 - \nu_1(t))\, d\Lambda^\circ(t)$$

the aiming point z° satisfies condition (2.16).

4. Example. Let us consider the time-optimal problem for the stochastic system ($\omega(t)$ is a Wiener process, $M\omega(t) \equiv 0$, $M\omega^2(t) = t$)

$$dx_1 = x_2 dt, \quad dx_2 = u(t)\, dt + d\omega, |u| \leqslant 1$$

$$x_1(0) = 1, \; x_2(0) = 0, \; x_1^{(\beta)} = -1, \; x_2^{\beta} = 0{,}2$$

under the coordinate constraints

$$M \| P(x(t_\beta) - x^{(\beta)}) \|^2 \leqslant (0.1)^2 + {}^1/_3 t_\beta^3 + {}^1/_4 t_\beta$$

$$M \| Qx(t) \|^2 \geqslant {}^7/_{16} + {}^1/_3 t^3 + {}^1/_4 t$$

$$P = Q = \begin{Vmatrix} 1 & 0 \\ 0 & {}^1/_2 \end{Vmatrix}$$

This problem is equivalent to the time-optimal problem for the deterministic system

$$\vartheta_1^{\cdot} = \vartheta_2, \; \vartheta_2^{\cdot} = u, \; |u| \leqslant 1 \tag{4.1}$$

$$\vartheta_1(0) = 1, \; \vartheta_2(0) = 0, \; \vartheta_1^{(\beta)} = -1, \; \vartheta_2^{(\beta)} = 0.2$$

under the coordinate constraints

$$\| P(\vartheta(t_\beta) - \vartheta^{(\beta)}) \|^2 \leqslant (0.1)^2, \quad \| Q\vartheta(t) \|^2 \geqslant {}^7/_{16} \tag{4.2}$$

Condition (3.3) for the deterministic problem (4.1), (4.2) takes the form

$$\max_{l(t)} \min_{\Lambda(t)} \min_p \Bigg\{ \int_0^{t_\beta^\circ} \Bigg| \int_0^{t_\beta^\circ} l_1(t)(t-\tau)\, d\Lambda(t) + \int_\tau^{t_\beta^\circ} l_2(t) \frac{1}{2}\, d\Lambda(t) + \tag{4.3}$$

$$p_1(t_\beta^\circ - \tau) + p_2 \frac{1}{2} \Bigg| d\tau + 2p_1 - 0.1 p_2 + 0.1 \sqrt{p_1^2 + p_2^2} + \int_0^{t_\beta^\circ} l_1(t)\, d\Lambda(t) -$$

$$\frac{1}{4} \int_0^{t_\beta^\circ} (l_1^2(t) + l_2^2(t))\, d\Lambda(t) - \frac{7}{16} \operatorname{Var} \Lambda(t) \Bigg\} = 0$$

over all $l(\cdot) \in W$, $\| p \| + \operatorname{Var} \Lambda(t) = 2$.

Solving problem (4.3), we obtain

$$t_\beta^\circ = 2\sqrt{2}, \quad l_1^\circ\left(\frac{\sqrt{6}}{2}\right) = \frac{1}{2}, \quad l_2^\circ\left(\frac{\sqrt{6}}{2}\right) = -\frac{\sqrt{6}}{2}$$

$$l_1^\circ\left(2\sqrt{2} - \frac{\sqrt{6}}{2}\right) = -\frac{1}{2}, \quad l_2^\circ\left(2\sqrt{2} - \frac{\sqrt{6}}{2}\right) = -\frac{\sqrt{6}}{2}$$

$$\frac{d\Lambda^\circ(t)}{dt} = \frac{\sqrt{2} - 1}{\sqrt{2}}\, \delta\left(t - \frac{\sqrt{6}}{2}\right) + \frac{1}{\sqrt{2}}\, \delta\left(t - 2\sqrt{2} + \frac{\sqrt{6}}{2}\right)$$

$$p_1^\circ = 0, \quad p_2^\circ = 1$$

The minimal function $h^\circ(t) = s^\circ(t) B(t)$ is determined by the equalities

$$h^\circ(t) = \frac{1}{2}, \qquad 2\sqrt{2} - \frac{\sqrt{6}}{2} \leqslant t \leqslant 2\sqrt{2}$$

$$h^\circ(t) = \frac{t - \sqrt{2}}{2\sqrt{2}}, \qquad \frac{\sqrt{6}}{2} \leqslant t < 2\sqrt{2} - \frac{\sqrt{6}}{2}$$

$$h^\circ(t) = \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)t - \frac{1}{2}, \qquad 0 \leqslant t < \frac{\sqrt{6}}{2}$$

The optimal control is determined from the maximum principle (2.14)

$$u^\circ(t) = -1, \; 0 \leqslant t \leqslant \sqrt{2}$$

$$u^\circ(t) = 1, \; \sqrt{2} < t \leqslant 2\sqrt{2}$$

The optimal trajectory touches the constraints for $t_1 = \sqrt{6}/2$, $t_2 = 2\sqrt{2} - \sqrt{6}/2$.

The authors are deeply grateful to A. B. Kurzhanskii and M. I. Gusev for useful discussions of the paper.

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Translated by N. H. C.

UDC 62-50

ON THE l-EVASION OF CONTACT IN A LINEAR DIFFERENTIAL GAME

PMM Vol. 38, № 3, 1974, pp. 417-421
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(Received April 25, 1973)

We derive sufficient conditions for the l-evasion of contact in a linear differential game. The paper adjoins the investigations in [1 – 5].

1. We consider the problem of evasion of contact [1, 2] in a linear differential game [3] given by the equation

$$\dot{z} = Cz + f(u, v), \qquad u \in P, \qquad v \in Q \tag{1.1}$$

Here z is a vector in the n-dimensional Euclidean space R^n, C is a constant nth-order square matrix, u is the pursuit parameter, v is the escape parameter, P and Q are given compact subsets from R^n, $f(u, v)$ is a function continuous in all its variables on $P \times Q$. The terminal set M of game (1.1) is assumed to be a linear subspace of space R^n.